My research is in **geometric group theory**, an area devoted to solving algebraic problems by studying groups as geometric objects. This is primarily done through the Cayley graphs of the group. One must beware that a group can have many Cayley graphs, depending on the choice of generating set. However, so long as these generating sets are finite, the distances in any two Cayley graphs are equivalent up to a fixed multiplicative and additive constant; we say the Cayley graphs are **quasi-isometric**. By restricting ourselves to geometric properties of Cayley graphs that are invariant under quasi-isometry, we therefore obtain invariants of the group itself.

A particularly powerful quasi-isometry invariant is the notion of **hyperbolicity** of a group, first introduced by Gromov in two seminal papers [35, 36]. This property mimics classical hyperbolic geometry in a way that is applicable to any geodesic metric space, including Cayley graphs. Hyperbolic groups enjoy a wide variety of useful properties, and include finite groups, free groups, small cancellation groups, and fundamental groups of closed hyperbolic manifolds. .

There have been many efforts to generalise the notion of hyperbolicity in order to apply its rich toolset to broader classes of groups. Generalisations include **relatively hyperbolic groups**, introduced by Gromov and further developed by Farb and many others [35, 30, 17, 21, 50], and **hierarchically hyperbolic groups** (HHGs), introduced by Behrstock, Hagen, and Sisto [10]. My research interests concern both of these, but I am primarily interested in developing the theory of HHGs. By extension, I am also interested in groups acting properly and cocompactly on CAT(0) **cube complexes**, most **3–manifold groups**, and **mapping class groups**, all of which are hierarchically hyperbolic and may therefore be studied using the powerful tools this provides [48, 49, 4, 13, 10, 12].

As these versions of hyperbolicity are all quasi-isometry invariants [25, 12], we know that any group quasi-isometric to a (relatively/hierarchically) hyperbolic group must itself be (relatively/hierarchically) hyperbolic. We say these classes of groups are **quasi-isometrically rigid**. We can then go on to ask how many quasi-isometry classes there are within these classes of groups, and what they look like. To answer this question fully, one must develop finer and finer invariants. Due to the vast diversity among groups, this is generally very difficult, and has motivated much research in geometric group theory. One major motivation of my research is working towards a quasi-isometric classification of HHGs.

My research programme can be summarised as follows.

(1) **New examples of hierarchically hyperbolic groups.**
- (1.1) *Almost HHGs.* In [BR1], Jacob Russell and I show that the structure of an "almost HHG" can always be promoted to a true HHG structure. This strengthens results of Abbott–Behrstock–Durham [1], and somewhat simplifies the definition of an HHG.
- (1.2) *Quotients of HHGs.* Abbott, Ng, Rasmussen, and I recently began a joint venture to show that certain quotients of HHGs are HHGs.
- (1.3) *Graph products and quasi-median graphs.* In [BR2], Russell and I study graph products by using quasi-median graphs, drawing on deep analogies with cube complexes. We show that graph products of HHGs are themselves HHGs, answering two questions of Behrstock–Hagen–Sisto. I aim to generalise this work by extending the existing theory of hierarchical hyperbolicity of cube complexes to quasi-median graphs.
- (1.4) *Graph braid groups.* In ongoing work, I construct an explicit HHG structure for graph braid groups by using a cubulation due to Abrams and Genevois [3, 32].

(2) **Applications of hierarchical hyperbolicity.**
- (2.1) *Applications to graph products.* Russell and I show that graph products are acylindrically hyperbolic and answer two questions of Genevois [BR2].
- (2.2) *Relative hyperbolicity and thickness.* A motivating goal is to classify HHGs up to quasi-isometry. One way to work towards this is by developing invariants such as thickness.
- (2.3) *Applications to graph braid groups.* My current work uses the cubical HHG structure of graph braid groups to classify their hyperbolicity, relative hyperbolicity, and thickness.
- (2.4) *JSJ decompositions of HHGs.* Work of Margolis uses JSJ decompositions of groups to give a partial quasi-isometric classification of right-angled Artin groups [47]. This suggests another potential route for classifying HHGs.

1. Hierarchically hyperbolic groups/spaces (HHGs/HHSs)

Hierarchical hyperbolicity is a quasi-isometry invariant property which combines elements of both Euclidean and hyperbolic geometry. This version of non-positive curvature was devised by Behrstock, Hagen, and Sisto by axiomatising Masur and Minsky's treatment of mapping class groups using subsurface projections and curve graphs [48, 49, 10]. As such, the geometric information of an HHS X is encoded in a collection of projections onto hyperbolic spaces associated to X. These projections are arranged via a partial order called **nesting**, and flats (copies of $\mathbb{Z}^n$) are encoded via a combinatorial relation between the projections called **orthogonality**. Due to the extra structure endowed by the projections and relations, one must be careful to distinguish a hierarchically hyperbolic *space* from a hierarchically hyperbolic *group*. An HHG is not merely a group whose Cayley graph is an HHS; the hierarchy structure must also be equivariant with respect to the group action.

Behrstock, Hagen, and Sisto showed that a wide range of groups/spaces are hierarchically hyperbolic [12], and used this unifying framework to further our understanding of their algebra and geometry [1, 8, 9]. Prominent examples include fundamental groups of Haglund and Wise's compact special cube complexes [39], the fundamental groups of closed 3–manifolds with no Nil or Sol components [12], mapping class groups [48, 49, 4, 13], and Teichmüller space with the Teichmüller or Weil–Petersson metric ([54, 26, 29] and [19, 4, 13] respectively). One of the overarching goals of my research is to introduce new classes of groups to the HHG arsenal. So far I have expanded the catalogue of HHGs by adding almost HHGs (Theorem 1), graph products (Theorem 5), and graph braid groups (Theorem 8) to this list [BR1, BR2]. These new examples are described in more detail below.

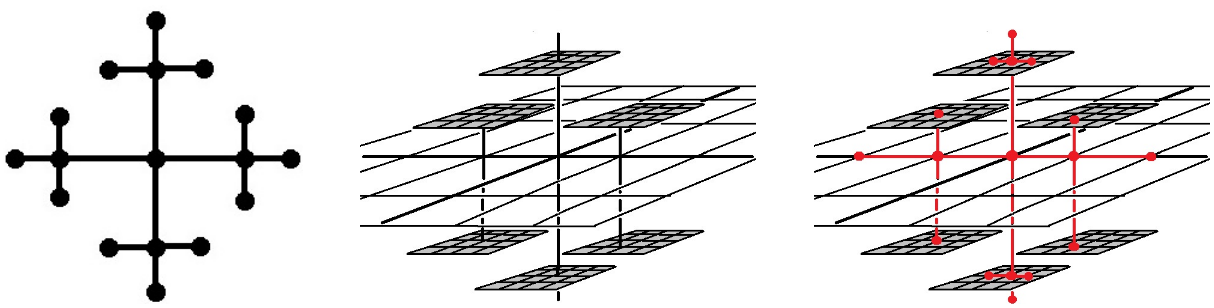

Figure 1. The right-angled Artin group $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$ can be endowed with an HHG structure. Its Cayley graph (centre) contains both Euclidean planes and copies of the Cayley graph of the hyperbolic group $\mathbb{Z} * \mathbb{Z}$ (left).

1.1. Almost HHGs.

Abbott, Behrstock, and Durham introduce the class of **almost HHGs** in [1], generalising HHGs by weakening one of the axioms. In a recent appendix to [1], Russell and I show that this *a priori* weaker structure of an almost HHG can in fact always be promoted to an HHG structure. This simplifies the definition of an HHG by allowing the weaker axiom to be used when proving hierarchical hyperbolicity of a group.

Theorem 1 ([BR1])**.** *Every almost HHG admits an HHG structure.*

This theorem strengthens results of Abbott–Behrstock–Durham about almost HHGs by showing that they also hold true in the general setting. In particular, combined with their results, this allows for a complete characterisation of contracting geodesics in HHGs.

Corollary 2 ([1])**.** *Let G be an HHG. For any $D > 0$ and $K \geqslant 1$ there exists $D' > 0$ depending only on D and G such that every (K, K)–quasigeodesic γ has the property that γ is D'–contracting if and only if γ has D–bounded projections.*

Theorem 1 also provides a crucial step in our proof that graph products of HHGs are HHGs (Theorem 5). I expect this result to have many future applications in verifying that a group is hierarchically hyperbolic.

1.2. **Quotients of HHGs.** I recently began work on a joint project with Carolyn Abbott, Thomas Ng, and Alexander Rasmussen, which I intend to continue during my postdoctoral position. We study **combinatorial HHGs**, a concept recently introduced by Behrstock, Hagen, Martin, and Sisto which provides a simple combinatorial criterion, in terms of an action on a hyperbolic simplicial complex, for a group to be an HHG [7]. Behrstock, Hagen, Martin, and Sisto apply this to show that quotients of mapping class groups by large powers of Dehn twists are HHGs. Motivated by a theorem of Delzant, which says that quotients of hyperbolic groups by high powers of elements with large asymptotic translation length are hyperbolic [23], we conjecture that a more general version of the Behrstock–Hagen–Martin–Sisto result exists.

Conjecture 3. Let G be an acylindrically hyperbolic combinatorial HHG. There exists a constant $N \geqslant 1$ such that for any infinite order element $g \in G$ and any non-zero integer k, the quotient $G/\langle\langle g^{kN} \rangle\rangle$ is a combinatorial HHG.

A result of Durham–Hagen–Sisto tells us that any infinite order element g in an HHG G is **axial** [27], that is, there exist associated hyperbolic spaces $C(U_1), \ldots, C(U_n)$ in the HHG structure in which the orbits of $\langle g \rangle$ are quasi-lines. Our idea is to fix a basepoint $x \in G$, consider its projections $\pi_i(x)$ to the hyperbolic spaces $C(U_i)$, and then cone off the axes $\langle g \rangle \cdot \pi_i(x)$ and their conjugates. If we then add in new line domains for each of the axes we coned off, this gives a new combinatorial HHG structure on G. We can then pass to the quotient to obtain a combinatorial HHG structure there.

In practice, the step of passing to the quotient is quite involved, with the Behrstock–Hagen–Martin–Sisto result relying on a generalisation of Dahmani's **composite rotating families** machinery to ensure lifts interact nicely with the combinatorial structure [21]. We expect that this is where most of the hard work will be. Additionally, we expect that we may need to impose a lower bound on the asymptotic translation length of our element g in the hyperbolic spaces $C(U_i)$, in analogy with Delzant's result [23].

Two immediate consequences of our conjecture would be that such quotients have finite asymptotic dimension and uniform exponential growth [9, 2]. This would be applicable to, for example, quotients of mapping class groups by large powers of reducible elements.

We also conjecture that quotients of combinatorial HHGs by generic elements are combinatorial HHGs, in the following sense.

Conjecture 4. Let G be an acylindrically hyperbolic combinatorial HHG. If μ is an admissible probability measure on G and w_n is a random walk of length n with respect to μ, then the quotient $G/\langle\langle w_n \rangle\rangle$ is a combinatorial HHG with probability approaching 1 as n tends to ∞.

A proposition of Dahmani, Guirardel, and Osin tells us that if a family of loxodromic elements of a group satisfies a small cancellation condition, then it is a rotating family [22]. Maher and Tiozzo show that given a random walk w_n with respect to an "admissible" probability measure μ, the family of its conjugates satisfies this small cancellation condition with probability approaching 1 as n tends to infinity [46]. As a result, we believe that we can use Behrstock, Hagen, Martin, and Sisto's rotating family techniques to show the quotient $G/\langle\langle w_n \rangle\rangle$ is a combinatorial HHG. This would provide an exciting new source of examples of HHGs.

1.3. **Graph products and quasi-median graphs.** Given a finite simplicial graph Γ and a collection of finitely generated groups $\{G_v\}$ indexed by the vertices of Γ, the **graph product** G_Γ is defined to be the free product of the vertex groups $\{G_v\}$ with commutation relations added between elements of G_v and G_w whenever v and w are connected by an edge in Γ. In particular, if the vertex groups are all copies of $\mathbb{Z}$, then G_Γ is the right-angled Artin group (RAAG) with defining graph Γ.

Behrstock, Hagen, and Sisto asked if graph products are HHGs when the vertex groups are HHGs. This was partially answered by Berlai and Robbio, giving a positive answer with some mild additional requirements on the vertex groups [16]. In joint work with Jacob Russell, we give a complete answer, showing that all graph products of HHGs are themselves HHGs.

Theorem 5 ([BR2])**.** *A graph product of HHGs is an HHG.*

We do this by first showing that graph products G_Γ are **relative HHGs**, that is, they are HHGs modulo the geometry of the vertex groups. This is achieved by replacing the word metric on G_Γ with the syllable metric, a technique originally used by Kim and Koberda in their study of RAAGs [44]. Roughly, this metric can be viewed as the equivalent of the Weil–Petersson metric on Teichmüller space. This new metric gives G_Γ the structure of a quasi-median graph, whose geometry has deep analogues with that of cube complexes, as explored by Genevois [31]. This cubical-like geometry allows us to adapt techniques used by Behrstock, Hagen, and Sisto in showing hierarchical hyperbolicity of RAAGs.

We then show that if the vertex groups are themselves HHGs, then we can extend the relative HHG structure through the vertex groups to obtain an HHG structure on G_Γ that uses the word metric. As a byproduct of this method, we answer a second question of Behrstock, Hagen, and Sisto, who asked if graph products are HHSs when endowed with the syllable metric.

Theorem 6 ([BR2])**.** *A graph product with the syllable metric is a hierarchically hyperbolic space.*

Our use of Genevois' theory of quasi-median graphs provokes more general questions. Quasi-median graphs have analogues of hyperplanes which generalise those of a cube complex, they admit projections onto convex subcomplexes, and moreover these analogues share many of the important geometric properties of their cubical counterparts. Since Behrstock, Hagen, and Sisto show that any cube complex with a **factor system** is an HHS, I conjecture that an equivalent notion exists for quasi-median graphs.

Question 7. Can an analogue of factor systems be developed for quasi-median graphs?

Genevois shows that certain wreath products, diagram products, and graphs of groups can be studied via quasi-median graphs [31], so this could possibly provide a new source of examples of HHGs.

1.4. **Graph braid groups.** Consider a finite collection of particles lying on a finite graph Γ. The **configuration space** of these particles on Γ is the collection of all possible ways the particles can be arranged on the graph with no two particles at the same point. As we move through the configuration space, the particles move along Γ, without colliding. If we do not distinguish each of the different particles, we call this an *unordered* configuration space. The **graph braid group** $B_n(\Gamma)$ is the fundamental group of the unordered configuration space of n particles on Γ.

The motion of the particles along Γ is continuous, thus the configuration space depends only on topological properties of Γ. By restricting the motion of particles to jumps between vertices, however, one can discretise this model, giving a combinatorial version of the configuration space. Work of Abrams and Genevois shows that this combinatorial configuration space is a compact special cube complex, and moreover, by adding sufficiently many valence 2 vertices along edges of Γ, Abrams shows that the topological configuration space deformation retracts onto the combinatorial one [3, 32]. Since valence 2 vertices do not affect the topology of Γ, this gives a method of cubulating any graph braid group. It then follows from work of Behrstock, Hagen, and Sisto that graph braid groups are HHGs [10].

Theorem 8. *Let Γ be a finite graph. Then $B_n(\Gamma)$ is an HHG for all $n \geqslant 1$.*

My current research involves constructing an explicit HHG structure for graph braid groups by translating the cubical structure into combinatorial properties of the graph. For example, by restricting to a subgraph Λ of Γ containing k of the n particles and fixing the remaining $n-k$ particles, one can find copies of the graph braid group $B_k(\Lambda)$ embedded as a subgroup of $B_n(\Gamma)$. By taking disjoint subgraphs Λ_1 and Λ_2 of Γ containing k_1 and k_2 particles respectively, with $k_1 + k_2 \leqslant n$, one obtains a direct product $B_{k_1}(\Lambda_1) \times B_{k_2}(\Lambda_2)$ embedded as a subgroup of $B_n(\Gamma)$. These correspond to nesting and orthogonality in the HHG structure, respectively. Knowing explicit aspects of the HHG structure provides more refined invariants for classification purposes; see Section 2.3 for my work in this direction.

2. Applications of hierarchical hyperbolicity

Hierarchically hyperbolic groups have far-reaching applications due to the rich diversity of groups they encompass. By studying HHGs in their full generality, it is possible to make advances in many classes of groups simultaneously. Moreover, the tools already available for one class may often be carried over to the others. For this reason, there is currently a great deal of activity in this rapidly expanding field of study. Exciting recent contributions include those provided by Dowdall–Durham–Leininger–Sisto [24], Durham–Minsky–Sisto [28], and Haettel–Hoda–Petyt [37], among others.

2.1. **Applications to graph products.** Russell and I apply our hierarchical hyperbolicity results for graph products to answer two questions of Genevois about the **electrification** of a graph product of finite groups. Graph products of finite groups form a particularly interesting class, as they include right-angled Coxeter groups (RACGs) and are the only cases where the syllable metric and word metric are quasi-isometric.

Genevois defines the electrification $\mathbb{E}(\Gamma)$ of a graph product G_Γ of finite groups by taking the syllable metric on G_Γ and adding edges between elements g, h whenever $g^{-1}h \in G_\Lambda \leqslant G_\Gamma$ and Λ is a **minsquare** subgraph of Γ, that is, a minimal subgraph that contains opposite vertices of a square if and only if it contains the whole square. Motivated by an analogy with relatively hyperbolic groups, Genevois proves that any quasi-isometry between graph products of finite groups induces a quasi-isometry between their electrifications, and uses this invariant to distinguish several quasi-isometry classes of RACGs [33]. The similarity of Genevois' definition of $\mathbb{E}(\Gamma)$ to the hyperbolic spaces in our own HHG structure allows us to leverage properties of the HHG structure to prove statements about $\mathbb{E}(\Gamma)$. In particular, we characterise when $\mathbb{E}(\Gamma)$ has bounded diameter, and when it is a quasi-line.

Theorem 9 ([BR2]). *Let G_Γ be a graph product of finite groups and let $\mathbb{E}(\Gamma)$ be its electrification.*

(1) $\mathbb{E}(\Gamma)$ has bounded diameter if and only if Γ is either a complete graph, a minsquare graph, or the join of minsquare graph and a complete graph.
(2) $\mathbb{E}(\Gamma)$ is a quasi-line if and only if G_Γ is virtually cyclic.

A key step in our proof of the second part of the theorem comes from the **acylindricity** of G_Γ afforded by its HHG structure [10]. A result of Osin then allows us to identify when G_Γ has independent loxodromic elements, which act as an obstruction to $\mathbb{E}(\Gamma)$ being a quasi-line [51].

Theorem 10 ([BR2]). *Let G_Γ be a graph product. The action of G_Γ by left multiplication on the maximal associated hyperbolic space in its HHG structure is acylindrical.*

2.2. **Relative hyperbolicity and thickness.** One of my motivating goals is to give a complete quasi-isometric classification of HHGs. This is unlikely to be achieved in the near future, but much progress has been made for specific classes of examples of HHGs. Closed 3–manifold groups have been almost completely classified, a culmination of the work of numerous authors [34, 43, 55, 53, 14, 15]. Mapping class groups are quasi-isometrically rigid in a strong sense: if a group is quasi-isometric to a mapping class group, then the two groups must be virtually equal [40]. Cubical groups remain elusive, although Huang and Margolis have made some recent progress towards classifying RAAGs [42, 41, 47]. I am working on contributing to the cubical case by classifying graph braid groups.

One way of producing results towards quasi-isometric classification is by developing fine quasi-isometry invariants. One such invariant is **thickness**, a concept introduced by Behrstock, Druţu, and Mosher as an obstruction to relative hyperbolicity [6]. Thickness comes in various orders, with each order being a quasi-isometry invariant. Thickness of order 0 is characterised by linear divergence of the group, and thickness of order k is defined inductively by stipulating that (a) the group is not thick of order $k-1$; and (b) any two points in the Cayley graph are connected by a sequence of thick pieces of order at most $k-1$ such that any two consecutive pieces have infinite intersection.

Roughly, thickness measures the complexity of coarse intersection patterns of non-negatively curved regions of the space. A result of Russell tells us that if collections of intersecting non-negatively curved regions in an HHG can be isolated from each other (the **isolated orthogonality** criterion), then the

HHG is relatively hyperbolic and these isolated collections form the peripheral subgroups [58]. In particular, the group is not thick. It is an open question whether all HHGs that are not relatively hyperbolic must be thick.

Conjecture 11. An HHG is thick if and only if it is not relatively hyperbolic.

There is already a lot of evidence towards this conjecture: the dichotomy has been proven for mapping class groups, 3–manifold groups, and Artin groups (Behrstock–Druţu–Mosher [6]), Teichmüller space (Brock–Masur [18]), Coxeter groups (Behrstock–Caprace–Hagen–Sisto [11]), and free-by-cyclic groups (Hagen [38]). My work with graph braid groups seeks to add a further example.

Conjecture 12. A graph braid group is thick if and only if it is not relatively hyperbolic.

2.3. **Applications to graph braid groups.** I am currently working on using the HHG structure of a graph braid group $B_n(\Gamma)$ to classify when it is hyperbolic, relatively hyperbolic, or thick. Moreover, I seek to frame this classification in terms of the structure of the graph Γ, giving a set of combinatorial criteria that can be easily checked. This aims to generalise work of Genevois, who obtained such criteria for toral relative hyperbolicity of graph braid groups [32].

To this end, I define the **orthogonality graph** $\mathcal{O}$ of $B_n(\Gamma)$, which has vertices corresponding to infinite-diameter copies of $B_k(\Lambda)$ embedded as subgroups of $B_n(\Gamma)$ and edges whenever $B_{k_1}(\Lambda_1)$ and $B_{k_2}(\Lambda_2)$ span a direct product $B_{k_1}(\Lambda_1) \times B_{k_2}(\Lambda_2)$ embedded as a subgroup of $B_n(\Gamma)$. Classifying hyperbolicity, relative hyperbolicity, and thickness of $B_n(\Gamma)$ then reduces to studying connectedness properties of $\mathcal{O}$. For example, the group $B_n(\Gamma)$ is hyperbolic if and only if $\mathcal{O}$ is totally disconnected, and Russell's isolated orthogonality criterion tells us that $B_n(\Gamma)$ is relatively hyperbolic if all connected components of $\mathcal{O}$ with non-zero diameter correspond to proper subgroups of $B_n(\Gamma)$ which are coarsely disjoint from each other.

In terms of Γ, the vertices of $\mathcal{O}$ correspond to subgraphs Λ of Γ containing cycles or stars, and Λ_1 and Λ_2 are connected by an edge in $\mathcal{O}$ if they are disjoint in Γ. Hyperbolicity of $B_n(\Gamma)$ therefore corresponds to a lack of disjoint pairs of cycle/star subgraphs of Γ, while relative hyperbolicity occurs when each component of $\mathcal{O}$ with non-zero diameter is supported on a proper subgraph of Γ and the intersections of these supports do not contain cycles/stars.

We may also characterise thickness in a similar manner. For example, $B_n(\Gamma)$ is thick of order 0 if Γ can be expressed as a disjoint union $\Gamma = \Lambda_1 \sqcup \Lambda_2$ where each Λ_i contains a cycle/star, and $B_n(\Gamma)$ is thick of order at most 1 if Γ can be covered by a sequence of subgraphs $\Lambda_1, \dots, \Lambda_m$ where each Λ_i contains a cycle/star and $\Lambda_i \cap \Lambda_{i+1} = \varnothing$ for each i. However, such characterisations only give an upper bound on the order of thickness, leaving open the question of precisely which orders of thickness may actually occur.

Question 13. Which orders of thickness can a graph braid group $B_n(\Gamma)$ have?

Determining precise orders of thickness is typically achieved by studying the **divergence** of the group, which is shown by Behrstock and Druţu to give a lower bound on order of thickness [5]. Taking inspiration from work of Levcovitz, it may therefore be fruitful to consider $\mathcal{O}$ as a **hypergraph** and study its "hypergraph index", which is shown to characterise divergence for RACGs [45]. I am currently exploring this avenue.

2.4. **JSJ decompositions of HHGs.** One important step in the quasi-isometric classification of 3–manifold groups is the JSJ decomposition, which splits the manifold along embedded tori into hyperbolic and Seifert-fibred pieces. Work of Gersten and Kapovich–Leeb shows that the presence of hyperbolic pieces can be quasi-isometrically detected, thus reducing the classification of irreducible non-geometric manifolds to the cases of graph manifolds and manifolds with hyperbolic pieces in their decomposition [34, 43]. This simplification of the problem then allowed Behrstock and Neumann to more or less complete the quasi-isometric classification [14, 15].

In a similar vein, Margolis gives a partial classification for one-ended RAAGs by using **JSJ decompositions of groups** [47], originally developed by Rips and Sela and further studied by many other authors as a generalisation of the 3–manifold case [56]. A JSJ decomposition of a group G is a finite graph of groups that encodes all possible splittings of G over subgroups in some fixed family $\mathcal{A}$; Margolis uses the family of two-ended subgroups. A result of Papasoglu implies that any quasi-isometry between one-ended finitely presented groups induces a graph isomorphism of the JSJ trees of cylinders, and moreover the vertices of the trees are categorised into three distinct types, each of which must be preserved under the quasi-isometry [52]. Further, the peripheral structure of each vertex stabiliser (that is, the set of conjugates of incident edge stabilisers) must also be preserved. This greatly simplifies the problem of quasi-isometric classification. Combining this with results of Cashen–Martin [20], Margolis is then able to give a classification for one-ended RAAGs that split over cyclic subgroups.

The above results of Papasoglu and Cashen–Martin apply in greater generality than simply RAAGs; they apply to all one-ended groups with two-ended splittings (with a mild technical condition for the Cashen–Martin result). There is therefore scope to adapt Margolis' techniques for use on some larger subclass of HHGs. Margolis' work also relies on being able to visually determine the JSJ tree of cylinders of a RAAG from its defining graph: given a defining graph Γ, one can construct a graph of groups $G(\Gamma)$ whose Bass–Serre tree is the JSJ tree of cylinders of the RAAG $A(\Gamma)$, and where every vertex and edge group of $G(\Gamma)$ is itself a RAAG. I conjecture that the HHG structure can be used to fulfil this role. As such, the vertex and edge groups should be HHGs. Following Berlai and Robbio's criteria for a finite graph of groups to produce an HHG [16], we may also expect the edge groups to embed as hierarchically quasiconvex subgroups.

Question 14. Can Margolis' techniques be generalised to a larger subclass of HHGs, thus providing a quasi-isometric classification of these groups?

To answer this question, one must identify precisely what the subclass is. We know that the HHGs in this subclass must be one-ended with two-ended splittings, and we expect that the vertex groups of such a splitting should themselves be HHGs. In order to better understand which HHGs split in this way, we can look to results of Robbio and Spriano, who characterise when hyperbolic-2-decomposable groups are HHGs [57]. Since the HHGs in our classification result should be (hierarchically hyperbolic)-2-decomposable, a good first step would be to generalise the Robbio–Spriano result. In fact, the authors pose this exact question in their paper, indicating that they expect much of their machinery to hold up in this more general setting.

Question 15. Which HHGs are (hierarchically hyperbolic)-2-decomposable?

Publications and preprints by Daniel Berlyne

[BR1] Daniel Berlyne and Jacob Russell. *Appendix to* Largest acylindrical actions and stability in hierarchically hyperbolic groups. *To appear in Transactions of the AMS*, 2020. Primary article by Carolyn Abbott, Jason Behrstock, and Matthew Gentry Durham.

[BR2] Daniel Berlyne and Jacob Russell. Hierarchical hyperbolicity of graph products. *arXiv:2006.03085*, 2020.

Other cited publications and preprints

[1] C. Abbott, J. Behrstock, and M. G. Durham. Largest acylindrical actions and stability in hierarchically hyperbolic groups. *To appear in Transactions of the AMS*, 2020. With an appendix by Daniel Berlyne and Jacob Russell.

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